STABILITY OF THE L^p-NORM OF THE CURVATURE TENSOR AT KÄHLER SPACE FORMS

SOMA MAITY

Abstract. We consider the Riemannian functional defined on the space of Riemannian metrics with unit volume on a closed smooth manifold M given by $\mathcal{R}_p(g) := \int_M |R(g)|^p dv_g$ where $R(g)$, dv_g denote the corresponding Riemannian curvature, volume form and $p \in [2, \infty)$. We prove that $\mathcal{R}_p$ restricted to the space of Kähler metrics attains its local minima at a metric with constant holomorphic sectional curvature.

1. Introduction

Let M be a closed smooth manifold of dimension $n \geq 3$ and $\mathcal{M}_1$ be the space of Riemannian metrics with unit volume on M endowed with the $C^{2,\alpha}$-topology for any $\alpha \in (0,1)$. In this paper we study the following Riemannian functional;

$$\mathcal{R}_p(g) = \int_M |R(g)|^p dv_g$$

where $R(g)$ and dv_g denote the corresponding Riemannian curvature tensor and volume form, $p \in [2, \infty)$. Let $\mathcal{M}_k$ denote the space of Kähler metrics with unite volume on M. Consider $C^{2,\alpha}$-topology on it.

Theorem. *Let (M, g) be a closed Kähler manifold with constant holomorphic sectional curvature and $p \in [2, \infty)$. Then (M, g) is a strict local minimizer for $\mathcal{R}_p$ restricted to $\mathcal{M}_k$ i.e. there exists a neighborhood $\mathcal{U}$ of g in $\mathcal{M}_k$ such that for any $\tilde{g} \in \mathcal{U}$*

$$\mathcal{R}_p(\tilde{g}) \geq \mathcal{R}_p(g).$$

The equality holds if and only if there exists a biholomorphism ϕ of M such that $\tilde{g} = \phi^* g$.

Remark : If (M, g) is a Kähler manifold with constant negative holomorphic sectional curvature then there are finite number of biholomorphisms of M. Hence there exists a neighborhood $\mathcal{U}_1$ of g in $\mathcal{M}_k$ such that for any $\tilde{g} \in \mathcal{U}_1$, $\mathcal{R}_p(\tilde{g})$ is strictly greater than $\mathcal{R}_p(g)$.

Definition : Let (M, g) be a Kähler manifold. A symmetric 2-tensor h on M is called a *Kähler* variation of g if there exists a one-parameter family of Kähler metrics $g(t)$ on M with $g(0) = g$ and $\frac{d}{dt} g(t)_{|t=0} = h$.

Let $\mathcal{K}$ denote the space of Kähler variations of g and $\mathcal{V}$ denote the space of symmetric 2-tensors orthogonal to the tangent space of the orbit of g under the action of the group of diffeomorphisms of M at g. First we prove that the Hessian of $\mathcal{R}_p$ has a positive lower bound when it is restricted to the space of unit vectors in $\mathcal{K} \cap \mathcal{V}$. The gradient of $\mathcal{R}_p$ is given by

$$\nabla \mathcal{R}_p = -p\delta^D D^*(|R|^{p-2}R) - p|R|^{p-2}\check{R} + \frac{1}{2}|R|^p g + (\frac{p}{n} - \frac{1}{2})\|R\|^p g$$

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For the notations we refer to Section 2. Every closed irreducible symmetric space is a critical point for $\mathcal{R}_p$. The Hessian at a critical point of $\mathcal{R}_p$ is given by

$$H(h_1, h_2) = \langle (\nabla \mathcal{R}_p)'_g(h_1), h_2 \rangle \quad \forall\ h_1, h_2 \in S^2(T^*M)$$

where $S^2(T^*M)$ denotes the space of symmetric 2-tensor fields on M and $(\nabla \mathcal{R}_p)'_g(h_1)$ denotes the derivative of $\nabla \mathcal{R}_p$ at g along h_1. A Riemannian metric g is called *rigid* if the kernel of H restricted to $\mathcal{V} \times \mathcal{V}$ is zero and it is called *stable* if H restricted to the products of unite vectors of $\mathcal{V}$ has a positive lower bound. It is difficult to investigate rigidity and stability of $\mathcal{R}_p$ for any arbitrary irreducible symmetric space. The strict stability of $\mathcal{R}_p$ has been obtained for Riemannian manifolds with constant sectional curvature and their suitable products for certain values of p [SM].

Let δ_g denote the divergence operator acting on $S^2(T^*M)$. The main result of this paper is the following.

Proposition. *Let (M, g) be a Riemannian manifold with constant holomorphic sectional curvature c. Then there exists a positive constant $k(c, p)$ such that for any $h \in \mathcal{K}$ with $tr(h) = 0$ and $\delta_g(h) = 0$.*

$$H(h, h) \geq k\|h\|^2$$

where $\|.\|$ denotes the L^2-norm on $S^2(T^*M)$ defined by g. The condition $\delta_g(h) = 0$ and $tr(h) = 0$ implies that $h \in \mathcal{K} \cap \mathcal{V}$ ([AB] Lemma 4.57).

Remark: For any $p \in [2, \frac{n}{2})$, if the first eigenvalue of the Laplacian of a compact hyperbolic manifold satisfies a lower bound then it is a local minimizer for $\mathcal{R}_p$ [SM]. In the spherical case, this holds for all $p \in [2, \infty)$. The extra condition in case of a compact hyperbolic manifold is required to prove the stability of $\mathcal{R}_p$ restricted to the conformal metrics. Since the only Kähler metrics in the conformal class of g are constant multiples of g, no extra condition is required to prove stability of $\mathcal{R}_p$ restricted to the Kähler variations.

Next we prove the existence of a local minimizing neighborhood of g. The idea of proof is based on the work of Gursky and Viaclovsky in [GV]. Some crucial observations about the space of Kähler variations and Kähler metrics are required here which are given in Lemma 4 and 5. Finally we end up with the following question.

Question : Is $\mathcal{R}_p$ stable for any compact Hermitian symmetric space?

2. Proof

2.1. **Preliminaries:** Let $\{e_i\}$ be an orthonormal basis at a point of M. $\check{R}$ is a symmetric 2-tensor defined by $\check{R}(x, y) = \sum R(x, e_i, e_j, e_k)R(y, e_i, e_j, e_k)$.
Define $\overset{\circ}{R}(h) : S^2(T^*M) \to S^2(T^*M)$ by

$$\overset{\circ}{R}(h)(x, y) := \sum R(e_i, x, e_j, y)h(e_i, e_j)$$

Let D and D^* be the Riemannian connection and its formal adjoint. Let $\Gamma(V)$ denote the space of sections of a vector bundle V and Λ^p denote the space of p-forms on M . Next we define some differential operators.

$d^D : S^2(T^*M) \to \Gamma(T^*M \otimes \Lambda^2 M)$ and its formal adjoint δ^D are defined by

$$d^D\alpha(x,y,z) := (D_y\alpha)(x,z) - (D_z\alpha)(x,y)$$
$$\delta^D(A)(x,y) = \sum\{D_{e_i}A(x,y,e_i) + D_{e_i}A(y,x,e_i)\}$$

The divergence operator δ_g on $S^2(T^*M)$ and its formal adjoint δ_g^* are defined by

$$\delta_g(h)(x) = -D_{e_i}h(e_i,x)$$
$$\delta_g^*\omega(x,y) := \frac{1}{2}(D_xy + D_yx)$$

Let g_t be a one-parameter family of metrics with $\frac{d}{dt}(g_t)_{|t=0} = h$ and $T(t)$ be a tensor depending on g_t. Then $\frac{d}{dt}T(t)_{|t=0}$ is denoted by $T'_g(h)$. Define $\Pi_h(x,y) = \frac{d}{dt}D_xy_{|t=0}$ and $\bar{r}_h(x,y) := R'_g(h)(x,e_i,y,e_i)$ where x,y are two fixed vector fields. The suffix h will be omitted when there will not be any ambiguity.
W is a 3-tensor defined by,

$$\begin{aligned} W_h(x,y,z): \;&=\; (D^*)'(h)(R)(x,y,z)\\ &-\; \sum[R(y,z,\Pi_h(e_i,e_i),w) + R(y,z,e_i,\Pi_h(e_i,w)) + R(z,e_i,\Pi_h(y,e_i),w)\\ &+\; R(z,e_i,e_i,\Pi_h(y,w)) + R(e_i,y,\Pi_h(z,e_i),w) + R(e_i,y,e_i,\Pi_h(z,w))]\end{aligned}$$

Next we prove some lemma which will be used in the proof of the proposition. Let (M,g) be a closed irreducible locally symmetric space and λ be its Einstein constant. Let h be a symmetric two tensor field such that $\delta_g(h) = 0$ and $tr(h) = 0$. Then from the equation (4.1) in [SM] we have,

$$H(h,h) = p|R|^{p-2}[\langle \bar{r}_h, \delta^D d^D h\rangle + \langle W_h, d^D h\rangle - \langle (\tilde{\mathcal{R}}_p)'(h), h\rangle + \frac{|R|^2}{n}\|h\|^2] \tag{2.1}$$

Next we compute each terms appeared in the above expression. We denote $T(e_{i_1}, e_{i_2}, ..., e_{i_k})$ by $T_{i_1 i_2 \dots i_k}$ where T is a k-tensor. Let $h \in S^2(T^*M)$ with $\delta_g(h) = 0$ and $tr(h) = 0$. Then we have the following lemma.

Lemma 1. $\bar{r}_h = \frac{1}{2}(D^*Dh + 2\lambda h)$

Proof. From [AB] 1.174(c), we have,

$$2(R'_g(h))_{piqi} \;=\; [(D^2_{iq}h)_{pi} + (D^2_{pi}h)_{qi} - (D^2_{pq}h)_{ii} - (D^2_{ii}h)_{pq} + h_{ij}R_{piqj} - h_{qj}R_{piij}]$$

Therefore,

$$\begin{aligned}\bar{r}_h(x,y) \;&=\; \frac{1}{2}\sum_i[D^2_{e_i,y}h(x,e_i) + D^2_{x,e_i}h(y,e_i) - D^2x,yh(e_i,e_i) - D^2_{e_i,e_i}h(x,y)\\ &\quad + h(R(x,e_i,y),e_i) - h(R(x,e_i,e_i),y)]\end{aligned}$$

Applying Ricci identity we have,

$$D^2_{e_i,y}h(x,e_i) - D^2_{y,e_i}h(e_i,x) = h(R(e_i,y,x),e_i) + h(R(e_i,y,e_i),x)$$

Notice that

$$\sum_i D^2_{y,e_i}h(e_i,x) = -D\delta_g h(y,x) = 0$$

$$\sum_i h(R(e_i,y,x),e_i) = -\overset{\circ}{R}(x,y)$$

and

$$\sum_i h(R(e_i, y, e_i), x) = \sum_j r(y, e_j)h(x, e_j) = \lambda h(x, y).$$

We also have,

$$\sum_i D^2_{e_i,e_i} h(x, y) = -D^* Dh(x, y)$$

and

$$\sum_i D^2_{x,y} h(e_i, e_i) = Ddtr(h) = 0$$

Combining all these results the lemma follows. □

Lemma 2. $\delta^D d^D h = 2D^* Dh + 2\lambda h - 2 \overset{\circ}{R}(h)$

Proof. From the identity 2.8 in [MB] we have,

$$\begin{aligned}\delta^D d^D h_{pq} &= 2D^* Dh_{pq} - 2\delta^*_g \delta_g h_{pq} + \sum_i (r_{pi} h_{iq} + r_{qi} h_{ip}) - 2\sum_{i,j} R_{piqj} h_{ij} \\ &= 2D^* Dh_{pq} + 2\lambda h_{pq} - 2 \overset{\circ}{R}(h)_{pq}\end{aligned}$$

□

Lemma 3. $\delta^D W = 2[\lambda D^* Dh + \lambda^2 h + D^* D(\overset{\circ}{R}(h)) - \overset{\circ}{R}(\overset{\circ}{R}(h))]$

Proof. From the proof of Lemma 4.1 (ii) in [SM] we have,

$$(W, d^D h) = 2\sum (R_{ij\Pi_{ik}l} - R_{li\Pi_{ik}j} - R_{lii\Pi_{kj}})(d^D h)_j kl$$

Now,

$$\begin{aligned}\sum R_{lii\Pi_{kj}} d^D h_{jkl} &= \sum C_{kjm} R_{liim} d^D h_{jkl} \\ &= -\sum r_{lm} C_{kjm} d^D h_{jkl} \\ &= -\lambda \sum C_{kjm} d^D h_{jkm} \\ &= -\frac{\lambda}{2} \sum [C_{kjm} - C_{mjk}] d^D h_{jkm} \\ &= -\frac{\lambda}{2} d^D h_{jkm} d^D h_{jkm}\end{aligned}$$

Therefore,

$$-2\int_M \sum R_{lii\Pi_{kj}} d^D h_{jkl} dv_g = \|d^D h\|^2$$

Using the previous lemma we have,

$$-2\int_M \sum R_{lii\Pi_{kj}} d^D h_{jkl} dv_g = 2[\lambda \|Dh\|^2 + \lambda^2 \|h\|^2 - \lambda \langle h, \overset{\circ}{R}(h)\rangle]$$

Now,

$$\sum (R_{ij\Pi_{ik}l} - R_{li\Pi_{ik}j}) d^D h_{jkl} = \frac{1}{2} \sum D_k h_{mi} (R_{ijml} - R_{limj}) d^D h_{jkl}$$

Since $DR = 0$, $\sum_{m,i} D_k h_{mi} R_{ijml} = D_k \overset{\circ}{R}(h)_{jl}$.
Therefore,

$$\begin{aligned}\sum (R_{ij\Pi_{ik}l} - R_{li\Pi_{ik}j}) d^D h_{jkl} &= \sum D_k \overset{\circ}{R}(h)_{jl} d^D h_{jkl} \\ &= (D \overset{\circ}{R}(h), Dh) - D_k \overset{\circ}{R}(h)_{jl} D_l h_{jk}\end{aligned}$$

Applying integration by parts,

$$\int_M D_k \overset{\circ}{R}(h)_{jl} D_l h_{jk} dv_g = -\int_M \overset{\circ}{R}(h)_{jl} D^2_{ij} h_{il} dv_g.$$

Since $\delta_g h = 0$, using Ricci identity,

$$\begin{aligned}\sum [D^2_{ij} h_{il} - D^2_{ji} h_{il}] &= \sum [h_{ml} R_{ijim} + h_{mi} R_{ijlm}] + D\delta_g h_{jl} \\ &= \lambda h_{jl} - \overset{\circ}{R}(h)_{jl}\end{aligned}$$

Therefore,

$$\langle \delta^D W, h\rangle = 2[\lambda \|Dh\|^2 + \lambda^2 \|h\|^2 + \langle \overset{\circ}{R}(h), D^* Dh\rangle - \| \overset{\circ}{R}(h)\|^2]$$

Hence the lemma follows. □

Combining Lemma 1-3 and using (2.1) we obtain that if (M, g) is an irreducible symmetric space and h is a symmetric two tensor with $\delta_g(h) = 0$ and $tr(h) = 0$ then

$$\begin{aligned}H(h,h) &= p|R|^{p-2}\big[\|D^*Dh\|^2 + \lambda\|Dh\|^2 - 3\langle \overset{\circ}{R}(h), D^*Dh\rangle + \frac{|R|^2}{n}\|h\|^2 \\ &\quad -2\lambda\langle h, \overset{\circ}{R}(h)\rangle + 2\| \overset{\circ}{R}(h)\|^2\big] - p|R|^{p-2}\langle (\check{R})'(h), h\rangle\end{aligned} \tag{2.2}$$

□

Next we will study H on the space of Kähler variations. For the definition of Kähler variation we refer to the introduction. A Kähler variation h is characterized the following two equations.

(k1) $h(Jx, Jy) = h(x, y)$

(k2) $J(\Pi_h(x, y)) = \Pi_h(x, Jy)$

Consider a closed Kähler manifold (M, g) with constant holomorphic sectional curvature. We can choose an orthonormal basis of the form $\{e_1, Je_1, ..., e_m, Je_m\}$. With respect to this basis R is given by

$R(e_i, e_j, e_k) = R(Je_i, Je_j, Je_k) = R(e_i, Je_j, e_k) = 0$ if $k \notin \{i, j\}$.

$R(e_i, e_j, e_i, e_j) = R(e_i, Je_j, e_i, Je_j) = R(Je_i, Je_j, Je_i, Je_j) = c$.

$R(e_i, Je_i, e_j, Je_j) = 2c$ for $i \neq j$.

$R(e_i, Je_i, e_i, Je_i) = 4c$ where c is a real number.

A simple calculation shows that the Einstein constant $\lambda = 2(m+1)c$, $|R|^2 = 32m(m+1)c^2$ and $\overset{\circ}{R}(h) = 2ch$ if $tr(h) = 0$.

2.2. **Proof of the proposition :** To prove the proposition using the formula (2.2) one needs to compute the term $\langle (\check{R}_g)'(h), h\rangle$. Let h be a Kähler variation with $tr(h) = 0$ and $\delta_g(h) = 0$.

$$\check{R}_{pq} = g^{i_1 i_2} g^{j_1 j_2} g^{k_1 k_2} R_{p i_1 j_1 k_1} R_{q i_2 j_2 k_2}$$

Differentiating each terms and evaluating it in an orthonormal basis $\{v_i\}$ and using

$$(g^{ij})' = -g^{im} h_{mn} g^{nj}$$

we have,

$$\begin{aligned}(\check{R}_g)'(h)_{pq} &= -h_{mn}\left(R_{pmij}R_{qnij} + R_{pimj}R_{qinj} + R_{pijm}R_{qijn}\right)\\ &\quad +(R'_g.h)_{pijk}R_{qijk} + R_{pijk}(R'_g.h)_{qijk}\end{aligned}$$

For detail computation we refer to Lemma 4.1 (i) in [SM]. Define a $(4,0)$ curvature tensor Q as

$$Q(v_q, v_i, v_j, v_k) = D^2_{v_i,v_j} h(v_q, v_k) + D^2_{v_q,v_k} h(v_i, v_j) - D^2_{v_q,v_j} h(v_i, v_k) - D^2_{v_i,v_k} h(v_q, v_j)$$

Hence

$$R'_g(h)(v_q, v_i, v_j, v_k) = \frac{1}{2}Q(v_q, v_i, v_j, v_k) + \frac{1}{2}\sum_m [h(v_k, v_m)R(v_q, v_i, v_j, v_m) - h(v_m, v_j)R(v_q, v_i, v_k, v_m)]$$

Next define a $(2,0)$ tensor,

$$S(v_p, v_q) = \frac{1}{2}\sum_{i,j,k} R(v_p, v_i, v_j, v_k)Q(v_q, v_i, v_j, v_k)$$

Therefore,

$$\begin{aligned}((\check{R}_g)'(h), h) &= -\sum h_{pq}h_{mn}\left(R_{pmij}R_{qnij} + R_{pimj}R_{qinj} + R_{pijm}R_{qijn}\right)\\ &+ 2(h, S) + \sum h_{pq}h_{mk}R_{pijk}R_{qijm}\end{aligned}$$

Next we switch to a basis of the form $\{e_1, Je_1, e_2, Je_2, ..\}$ to use the nice form of the curvature tensor.

$$\begin{aligned}S(e_p, e_q) &= \sum_i R(e_p, e_i, e_p, e_i)Q(e_p, e_i, e_p, e_i) + \sum_{p\neq i} R(e_p, Je_i, e_p, Je_i)Q(e_q, Je_i, e_p, Je_i)\\ &\quad + \sum_{p\neq i} R(e_p, e_i, Je_p, Je_i)Q(e_q, e_i, Je_p, Je_i) + \sum_{p\neq i} R(e_p, Je_i, Je_p, e_i)Q(e_p, Je_i, Je_p, e_i)\\ &\quad + \sum_{p\neq i} R(e_p, Je_p, e_i, Je_i) + R(e_p, Je_p, e_p, Je_p)Q(e_q, Je_p, e_p, Je_p)\\ &= c\sum_{i\neq p}[Q(e_q, e_i, e_p, e_i) + Q(e_q, Je_i, e_p, Je_i) + Q(e_q, e_i, Je_p, Je_i) - Q(e_q, Je_i, Je_p, e_i)\\ &\quad +2Q(e_q, Je_p, e_i, Je_i)] + 4cQ(e_p, Je_p, e_p, Je_p)\end{aligned}$$

Similarly we have,

$$\begin{aligned}S(e_p, Je_q) &= c\sum_{i\neq p}[Q(Je_q, e_i, e_p, e_i) + Q(Je_q, Je_i, e_p, Je_i) + Q(Je_q, e_i, Je_p, Je_i)\\ &\quad -Q(Je_q, Je_i, Je_p, e_i) + 2Q(Je_q, Je_p, e_i, Je_i)] + 4cQ(Je_q, Je_p, e_p, Je_p)\end{aligned}$$

$$\begin{aligned} S(Je_p, e_q) &= c\sum_{i\neq p}[Q(e_q, Je_i, Je_p, Je_i) + Q(Je_q, e_i, Je_p, e-i) + Q(e_q, Je_i, e_p, e_i) \\ &\quad -Q(Je_q, e_i, e_p, Je_i) + 2Q(e_q, e_p, Je_i, e_i)] + 4cQ(e_q, e_p, Je_p, e_p) \end{aligned}$$

$$\begin{aligned} S(Je_p, Je_q) &= c\sum_{i\neq p}[Q(Je_q, Je_i, Je_p, Je_i) + Q(Je_q, e_i, Je_p, e_i) + Q(Je_q, Je_i, e_p, e_i) \\ &\quad -Q(Je_q, e_i, e_p, Je_i) + 2Q(Je_q, e_p, Je_i, e_i)] + 4cQ(Je_q, e_p, Je_p, e_p) \end{aligned}$$

Define r_Q by trace of Q in 2nd and 3rd entries. Therefore,

$$\begin{aligned} (S, h) &= c\sum h(e_p, e_q)[Q(e_q, e_i, Je_p, Je_i) - Q(e_q, Je_i, Je_p, e_i) + 2Q(e_q, Je_p, e_i, Je_i)] \\ &+ c\sum h(e_p, Je_q)[Q(Je_q, e_i, Je_p, Je_i) - Q(Je_q, Je_i, Je_p, e_i) + 2Q(Je_q, Je_p, e_i, Je_i)] \\ &+ c\sum h(Je_p, Je_q)[Q(Je_q, Je_i, e_p, e_i) - Q(Je_q, e_i, e_p, Je_i) + 2Q(Je_q, e_p, Je_i, e_i)] \\ &+ c\sum h(Je_p, e_q)[Q(e_q, Je_i, e_p, e_i) - Q(e_q, e_i, e_p, Je_i) + 2Q(e_q, e_p, Je_i, e_i)] \\ &+ c(r_Q, h) \end{aligned}$$

$$\begin{aligned} &\sum_i Q(e_q, Je_p, e_i, Je_i) \\ = &\sum_i [D^2_{Je_p, e_i} h(e_q, Je_i) + D^2_{e_q, Je_i} h(Je_p, e_i) - D^2_{e_q, e_i} h(Je_p, Je_i) - D^2_{Je_p, Je_i} h(e_q, e_i)] \end{aligned}$$

By a simple calculation we have,

$$D^2_{x,y} h(Jw, Jz) = D^2_{x,y} h(w, z) \tag{2.3}$$

Therefore,

$$\begin{aligned} \sum_i [D^2_{Je_p, e_i} h(e_q, Je_i) - D^2_{Je_p, Je_i} h(e_q, e_i)] &= \sum_i [-D^2_{Je_i, e_i} h(Je_q, e_i) - D^2_{Je_p, Je_i} h(Je_q, Je_i)] \\ &= 2\delta_g^* \delta_g h(Je_p, Je_q) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \sum_i [D^2_{e_q, Je_i} h(Je_p, e_i) - D^2_{e_q, e_i} h(Je_p, Je_i)] &= -\sum_i [D^2_{e_q, Je_i} h(e_p, Je_i) + D^2_{e_q, e_i} h(e_p, e_i)] \\ &= 2\delta_g^* \delta_g h(e_q, Je_p) \\ &= 0 \end{aligned}$$

Hence,

$$\sum_i Q(e_q, Je_p, e_i, Je_i) = 0$$

Similarly,

$$\sum_i Q(Je_q, Je_p, e_i, Je_i) = \sum_i Q(Je_q, e_p, Je_i, e_i) = \sum_i Q(e_q, e_p, Je_i, e_i) = 0$$

Next,

$$\begin{aligned}
& \sum_i [Q(Je_q, e_i, Je_p, Je_i) - Q(Je_q, Je_i, Je_p, e_i)] \\
= & \sum_i [D^2_{e_i, Je_p} h(Je_q, Je_i) + D^2_{Je_q, Je_i} h(e_i, Je_p) - D^2_{e_i, Je_i} h(Je_q, Je_p) \\
+ & D^2_{Je_i, e_i} h(Je_q, Je_p) - D^2_{Je_q, e_i} h(Je_i, Je_p) - D^2_{Je_i, Je_p} h(Je_q, e_i)]
\end{aligned}$$

Applying Ricci identity we have,

$$\begin{aligned}
& \sum_{i,j} [D^2_{Je_i, e_i} h(Je_q, Je_p) - D^2_{e_i, Je_i} h(Je_q, Je_p)] \\
= & \sum_{i,j} [h(Je_j, e_p) R(Je_i, e_i, e_q, Je_j) + h(Je_j, e_q) R(Je_i, e_i, e_p, Je_i) \\
= & \sum_i [h(Je_q, e_p) R(Je_i, e_i, e_q, Je_q) + h(Je_p, e_q) R(Je_i, e_i, e_p, Je_p)] \\
= & \ 0
\end{aligned}$$

$$\begin{aligned}
& \sum_i [D^2_{e_i, Je_p} h(Je_q, Je_i) - D^2_{Je_i, Je_p} h(Je_q, e_i)] \\
= & \sum_i [D^2_{e_i, Je_p} h(Je_q, Je_i) - D^2_{Je_p, e_i} h(Je_q, Je_i)] \\
+ & \sum_i [D^2_{Je_p, Je_i} h(Je_q, e_i) - D^2_{Je_i, Je_p} h(Je_q, e_i)] \\
= & \sum_{i,j} [h(Je_i, e_j) R(e_i, Je_p, Je_q, e_j) + h(Je_q, e_j) R(e_i, Je_p, Je_i, e_j)] \\
+ & \sum_{i,j} [h(e_i, Je_j) R(Je_p, Je_i, Je_q, Je_j) + h(e_j, Je_q) R(Je_p, Je_i, e_i, e_j)] \\
= & -2c(m+1) h(e_p, Je_q)
\end{aligned}$$

$$\begin{aligned}
\sum_i [D^2_{Je_q, Je_i} h(e_i, Je_p) - D^2_{Je_q, e_i} h(Je_i, Je_p)] &= -\sum_i [D^2_{Je_q, Je_i} h(Je_i, Je_p) + D^2_{Je_q, e_i} h(e_i, Je_p)] \\
&= \delta^*_g \delta_g h(Je_p, Je_q) \\
&= 0
\end{aligned}$$

Hence,

$$\sum_i [Q(Je_q, e_i, Je_p, Je_i) - Q(Je_q, Je_i, Je_p, e_i)] = -2c(m+1) h(e_p, Je_q)$$

Following similar computation we have,

$$\sum_i [Q(e_q, e_i, Je_p, Je_i) - Q(e_q, Je_i, Je_p, e_i)] = -2c(m+1) h(e_p, e_q)$$

$$\sum_i [Q(Je_q, Je_i, e_p, e_i) - Q(Je_q, e_i, e_p, Je_i)] = -2c(m+1) h(Je_p, Je_q)$$

and

$$\sum_i [Q(e_q, Je_i, e_p, e_i) - Q(e_q, e_i, e_p, Je_i)] = -2c(m+1) h(Je_p, e_q)$$

Hence

$$(S,h) = c(r_Q,h) - 2c^2(m+1)|h|^2$$

Next we will compute the term (r_Q,h).

$$\begin{aligned}
r_Q(e_p,e_q) &= \sum_i [Q(e_p,e_i,e_q,e_i) + Q(e_p,Je_i,e_q,Je_i)] \\
&= \sum_i [D^2_{e_i,e_q}h(e_p,e_i) + D^2_{e_p,e_i}h(e_i,e_q) + D^2_{Je_i,e_q}h(e_p,Je_i) + D^2_{e_p,Je_i}h(e_q,Je_i)] \\
&- \sum_i [D^2_{e_p,e_q}(e_i,e_i) + D^2_{e_i,e_i}h(e_p,e_q) + D^2_{e_p,e_q}h(Je_i,Je_i) + D^2_{Je_i,Je_i}h(e_p,e_q)]
\end{aligned}$$

$$\begin{aligned}
&\sum_i [D^2_{e_p,e_q}(e_i,e_i) + D^2_{e_p,e_q}h(Je_i,Je_i)] = Ddtrh(e_p,e_q) = 0 \\
&\sum_i [D^2_{e_i,e_i}h(e_p,e_q) + D^2_{Je_i,Je_i}h(e_p,e_q)] = -D^*Dh(e_p,e_q) \\
&\sum_i [D^2_{e_p,e_i}h(e_i,e_q) + D^2_{e_p,Je_i}h(e_q,Je_i)] = -2\delta^*_g\delta_g h(e_p,e_q) = 0
\end{aligned}$$

Now applying Ricci identity to the remaining terms after adding the term $2\delta^*_g\delta_g h = 0$, we have

$$\begin{aligned}
&\sum_i [D^2_{e_i,e_q}h(e_p,e_i) + D^2_{Je_i,e_q}h(e_p,Je_i)] \\
= &\sum_{i,j} [R(e_i,e_q,e_p,e_j)h(e_j,e_i) + R(e_i,e_q,e_i,e_j)h(e_j,e_p) \\
+ &R(Je_i,e_q,e_p,Je_j)h(Je_j,Je_i) + R(Je_i,e_q,Je_i,e_j)h(e_j,e_p)] \\
= &(\lambda+6c)h(e_p,e_q) \\
= &2c(m+4)h(e_p,e_q)
\end{aligned}$$

Hence

$$r_Q(e_p,e_q) = D^*Dh(e_p,e_q) + 2c(m+4)h(e_p,e_q)$$

Similarly computing the other coefficients of r_Q we get

$$r_Q = D^*Dh + 2(m+4)h$$

Therefore,

$$(S,h) = c|Dh|^2 + 6c^2|h|^2$$

Next a simple calculation gives,

$$\begin{aligned}
&-\sum h_{pq}h_{mn}\left(R_{pmij}R_{qnij} + R_{pimj}R_{qinj} + R_{pijm}R_{qijn}\right) + \sum h_{pq}h_{mn}R_{pijn}R_{qijm} \\
= &-\sum [h_{mn}h_{pq}R_{pijm}R_{qijn} + h_{mn}h_{pq}R_{pmij}R_{qnij}]
\end{aligned}$$

Since h, R are invariant under the action of J,

$$\begin{aligned}
& \sum h_{mn}h_{pq}R_{pmij}R_{qnij} \\
= & \ 4\sum h(e_p,e_q)h(e_m,e_n)[R(e_p,e_m,e_i,e_j)R(e_q,e_n,e_i,e_j) \\
+ & \ R(e_p,Je_m,e_i,Je_j)R(e_q,Je_n,e_i,Je_j)] \\
+ & \ 4\sum h(Je_p,e_q)h(Je_m,e_n)[R(e_p,e_m,e_i,e_j)R(e_q,e_n,Je_i,Je_j) \\
- & \ R(Je_p,e_m,Je_i,e_j)R(e_q,Je_n,Je_i,e_j)] \\
= & \ 16(m+1)c^2\sum[h^2(e_p,e_q)+h^2(Je_p,e_q)] \\
= & \ 8c^2(m+1)|h|^2
\end{aligned}$$

$$\begin{aligned}
& \sum h_{mn}h_{pq}R_{pijm}R_{qijn} \\
= & \ 4\sum h(e_p,e_q)h(e_m,e_n)[R(e_p,e_i,e_m,e_j)R(e_q,e_i,e_n,e_j) \\
+ & \ R(e_p,Je_i,e_m,Je_j)R(e_q,Je_i,e_n,Je_j)] \\
+ & \ 4\sum h(Je_p,e_q)h(Je_m,e_n)[R(Je_p,e_i,Je_m,e_j)R(e_q,e_i,e_n,e_j) \\
+ & \ R(e_q,e_i,e_m,e_j)R(e_q,Je_i,e_n,Je_j)] \\
= & \ 8c^2(m+1)\sum[h^2(e_p,e_q)+h^2(Je_p,e_q)] \\
= & \ 4c^2(m+1)|h|^2
\end{aligned}$$

Therefore,

$$\langle(\check{R})'(h),h\rangle = 2c\|Dh\|^2 - 12c^2m\|h\|^2$$

From the formula (2.2) we obtain,

$$\begin{aligned}
H(h,h) &= p|R|^{p-2}[\|D^*Dh\|^2 + 2c(m-3)\|Dh\|^2 + 4c^2m(4m+5)\|h\|^2] \\
&= p|R|^{p-2}[\|D^*Dh + (m-3)ch\|^2 + (15m^2+14m+6)c^2\|h\|^2] \\
&\geq k\|h\|^2
\end{aligned}$$

where k is a positive constant. Hence, the proof of the Proposition follows. $\square$

2.3. **Proof of the main Theorem:** Next we prove the existence of a local minimizing neighborhood around g. Let $\mathcal{H}$ denote the space of holomorphic vector fields on M. If (M,g) is a closed Kähler Einstein manifold then by a theorem by Lichnerowicz the dimension of $\mathcal{H}$ is finite. If (M,g) is a Kähler manifold with constant negative holomorphic sectional curvature then $\mathcal{H}=0$ [AB] Proposition 2.138.

Lemma 4. *Let (M,g) be a Kähler Einstein manifold with positive scalar curvature. Then*

$$\mathcal{K}_g = \mathbb{R}\oplus\delta_g^*(\mathcal{H})\oplus(\mathcal{K}_g\cap\delta_g^{-1}(0)\cap tr^{-1}(0)).$$

Proof. It is easy to see that $\delta_g^*(\mathcal{H})\subset\mathcal{K}_g$. Let $h\in\mathcal{K}_g$ and x,y,z are three vector fields. Then

$$\Pi_h(Jx,Jy) = -\Pi_h(x,y).$$

Hence

$$(\Pi_h(e_i,e_i),z) + (\Pi_h(Je_i,Je_i),z) = 0.$$

This implies,

$$D_z h(e_i, e_i) + D_z(Je_i, Je_i) = 2[D_{Je_i} h(z, Je_i) + D_{e_i} h(z, e_i)]$$

Therefore,

$$dtrh(z) + 2\delta_g h(z) = 0 \tag{2.4}$$

When $h = \delta_g^* \omega$ for some one form ω then

$$2\delta_g \delta_g^* \omega = d\delta\omega \tag{2.5}$$

Any one form ω also satisfies the following identity.

$$2\delta_g \delta_g^* \omega + \delta d\omega = 2D^* D\omega \tag{2.6}$$

Using Böchner technique on the space of one forms we have

$$\Delta\omega = D^* D\omega + \lambda\omega. \tag{2.7}$$

where λ denotes the Einstein constant. Lichnerowicz's criterion for holomorphic vector fields says that $\omega^\sharp$ is a holomorphic vector field if and only if

$$\frac{1}{2}\Delta\omega = \lambda\omega \tag{2.8}$$

Combining the equations (2.5)-(2.8) we have that $\delta_g^* \omega$ is a Kähler variation if and only if $\omega^\sharp$ is a holomorphic vector field. We also have that if $\omega^\sharp$ is a holomorphic vector field then so is $\delta_g \delta_g^* \omega$. (2.4) implies that if fg is a Kähler variation for some smooth function f on M then f is a constant function. Now the proof follows from the decomposition given in [AB] Lemma 4.57. □

The following lemma is analogous to the Lemma 2.10 in [GV].

Lemma 5. *For Kähler metric $\tilde{g} = g + \theta_1$ in a sufficiently small $C^{l+1,\alpha}$-neighborhood of the Kähler Einstein metric g ($l \geq 1$), there is a automorphism ϕ of M and a constant c such that*

$$\tilde{\theta} = e^c \phi^* \tilde{g} - g$$

satisfies, $\delta_g \tilde{\theta} = 0$ and $tr(\tilde{\theta}) = 0$

Proof. Consider the map $\mathcal{N} : \mathcal{H} \times \mathbb{R} \times \mathcal{K} \to \mathcal{H} \times \mathcal{R}$ given by,

$$\begin{aligned} \mathcal{N}(x,t,\theta) &= \mathcal{N}_\theta(x,t) \\ &= \big(\delta_g[\phi_{x,1}^*(g+\theta)], e^t \int_M tr_g[\phi_{x,1}^*(g+\theta)]dv_g - nV(g)\big) \end{aligned}$$

where $\phi_{x,1}$ denotes the diffeomorphism obtained by following the flow generated by the vector field x for unit time. Linearizing this map in (x,t) at $(0,0,0)$ we obtain,

$$\begin{aligned} \mathcal{N}_0'(y,s) &= (\delta_g L_g(y), snV(g)) \\ &= -\frac{1}{2}\delta_g \delta_g^*(y), snV(g)) \end{aligned}$$

where L denotes Lie derivative. It is easy to see that $\mathcal{N}'$ is surjective. By implicit function theorem, given a θ_1 small enough there exists $x \in \mathcal{H}$ and a real number t such that

$$\delta_g[\phi_{x,1}^*(g+\theta_1)] = 0 \quad \text{and} \quad e^t \int_M tr_g[\phi_{x,1}^*(g+\theta_1)] = nV(g)$$

Let $\phi = \phi_{x,1}$ and $\tilde{\theta} = e^t \phi^*(g+\theta_1) - g$. Then ϕ and $\tilde{\theta}$ satisfies the conditions given in the Lemma. □

Let $\mathcal{M}$ denote the space of Riemannian metrics on M, $\tilde{\mathcal{R}}_p$ denote the Riemannian functional defined by volume normalization of $\mathcal{R}_p$ on $\mathcal{M}$ and $\tilde{H}_g$ its 2nd derivative at g. Let $\mathcal{W} = (\mathcal{K}_g \cap \delta_g^{-1}(0) \cap tr^{-1}(0))$. To complete the proof of the main theorem we recall some results from [SM].

Let g be a Riemannian metric on M with unit volume. There exists a neighborhood U of g in $\mathcal{M}_1$ such that for any $\tilde{g} \in U$ and $h \in \mathcal{W}$,

$$|\tilde{H}_{\tilde{g}}(h,h) - \tilde{H}_g(h,h)| \leq C\|\tilde{g} - g\|^4_{C^{2,\alpha}}\|h\|^2_{L^{2,2}}$$

where

$$\|h\|^2_{L^{2,2}} = \|D^2h\|^2 + \|Dh\|^2 + \|h\|^2.$$

Using the Proposition one can prove that if (M,g) is a metric with constant holomorphic sectional curvature then there exists a positive constant k_1 such that for every $h \in (\mathcal{K}_g \cap \delta_g^{-1}(0) \cap tr^{-1}(0))$

$$\tilde{H}_g(h,h) \geq k_1\|h\|^2_{L^{2,2}}.$$

Now consider a neighborhood $\mathcal{U}$ of g in the space of Kählar metrics where Lemma 5 and 6 holds. Let $\tilde{g} \in \mathcal{U}$. Using Lemma 5 we have a Kähler metric g_0 in $\mathcal{U}$ such that $g - g_0 \in \mathcal{W}$. Now consider $\gamma(t) = tg_0 + (1-t)g$. Since $\tilde{H}_{\gamma(t)}$ restricted to the space $\mathcal{W}$ is positive definite integrating $\tilde{H}$ along $\gamma(t)$ we obtain the main theorem. For detail description we refer to the proof of Proposition 3 in [SM]. □

Remark : If (M,g) is a Kähler manifold with constant negative holomorphic sectional curvature then there are finitely many biholomorphisms of M. So the space of Kähler metrics intersect the orbit of group of diffeomorphisms of g in finitely many points. In this case Lemma 4 and 5 are not required to prove the main theorem.

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Department of Mathematics, Indian Institute of Science, Bangalore-12, India
E-mail address: somamaity@math.iisc.ernet.in